A parallel method for the numerical solution of integro-differential equation with positive memory

Kiwoon Kwon a, Dongwoo Sheen b,*

a Institute for Numerical and Applied Mathematics, George-August-University Göttingen, Lotzestrasse 16–18, D-37083 Göttingen, Germany
b Department of Mathematics, Seoul National University, Seoul 151-747, South Korea

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Abstract

An efficient parallel numerical method is proposed for an integro-differential equation with positive memory. Instead of solving the equation in classical time-marching methods which require massive storage of solutions of previous time steps in order to advance to a next time step, the Fourier–Laplace transformation in time is applied to obtain a set of complex-valued, elliptic problems parameterized by points on a contour in the complex plane. Using the independence of an elliptic problem corresponding to one contour point is independent of those elliptic problems corresponding to other contour points, all elliptic problems can be solved in parallel essentially without data communications. Then the time domain solution can be obtained by the Fourier–Laplace inversion formula. An error analysis and the numerical implementation of this parallel method is presented.

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1. Introduction

The principal aim of this paper is to describe a parallel numerical method for a parabolic integro-differential equation representing a heat conduction in material with positive memory. Classically, a heat conduction phenomenon is represented by a parabolic partial differential equation with an infinite heat propagation speed; this is a puzzling contradiction to the physics. Indeed, the material property of the past influences on that of the present, and therefore the heat propagation can be better understood if it is represented by an integro-differential equation rather than it is modeled by the usual parabolic equations. In this paper we consider the integro-differential equation:

\[ u_t(x,t) + \mu Au(x,t) + \int_0^t K(t-s)Bu(x,s)\,ds = f(x,t) \quad (x,t) \in \Omega \times [0, \infty), \]  

(1.1a)
\[ u(x, t) = 0 \quad (x, t) \in \partial \Omega \times [0, \infty), \quad (1.1b) \]
\[ u(x, 0) = u_0 \quad x \in \Omega \times \{t = 0\}, \quad (1.1c) \]

where \( K \in L^1(0, T) \) is a positive memory, \( A \) and \( B \) are positive-definite self-adjoint operator and second order operator with smooth coefficients in \( x \) and \( t \) in a real Hilbert space \( H \) with norm \( \| \cdot \|_H \), and \( \mu \) is a nonnegative real number.

Eq. (1.1) can be found in modeling heat flow through materials with memory [3,13,19,23] and in many viscoelastic problems [5]. If \( \mu > 0 \), (1.1) is parabolic, and if \( \mu = 0 \) and \( K(0) > 0 \), (1.1) becomes hyperbolic which can be easily seen by differentiating it with respect to \( t \) [35]. However, if \( \mu = 0 \) and \( K \) is singular at the origin, (1.1) can be regarded as intermediate between parabolic and hyperbolic. In addition that \( \mu = 0 \), if \( K \) is a positive constant, Eq. (1.1a) represents the wave equation with finite wave speed \( \sqrt{K} \) and thus an initial discontinuity propagates without being smoothed out. On the other hand, if \( \mu = 0 \) and \( K(\cdot) \) is the Dirac delta function \( \delta(\cdot) \), Eq. (1.1) reduces to the classical heat equation with smoothing effect. Therefore (1.1) has an intermediate property between the heat and wave equations and it possesses the double features that the speed of propagation is finite and the discontinuity can be smoothed out. Hrusa, Prüss, Desch and others proved that the speed of propagation is related to the assumption \( K(0) < \infty \) and that the discontinuities are smoothed out if \( K'(0^+) = -\infty \): see, for instance, [7,14,25,26].

Usually approximate solutions to this problem are calculated by time stepping methods such as backward Euler, Crank–Nicolson methods, and collocation methods by storing all or a part of the previous time step solutions to compute the integrals containing the memory \( K \). Several efficient numerical methods for (1.1) have been proposed and analyzed: for instance, see [4,6,8,11,18,20,21,24,27,33,36] and the references therein. Especially, if \( B \) is also a symmetric nonnegative definite operator, not very different from \( A \) in a certain sense, there have been more analyses including the relatively long time behavior [1,32]. In order to store previous time step solutions, these schemes need huge computational memories, especially if one wants to deal with three-dimensional problems. Moreover, these approaches are not naturally parallelized because of the dependence in memory term of the past time step solutions.

In this work, instead of following standard methods, we begin by applying the Laplace transformation in time to (1.1). Douglas et al. [9,10], introduced an efficient method for solving wave equations in the frequency domain after taking the Fourier transformation of the original problems; then after solving complex-valued elliptic problems for each frequency, the solutions in the time–space domain are obtained by the inversion formula. Among others one of the major benefits in this approach is that the Fourier-transformed elliptic problems can be solved in parallel for each frequency independently, which reduces communication costs significantly when actual parallel computational architectures are employed. Based on and inspired by the above work, the theory has been further developed and extended to cover viscoelasticity [15], parabolic problems [17], and linearized Navier–Stokes equations [16]. Later Sheen–Sloan–Thomeé applied the Laplace transformation instead of the Fourier transformation to solve parabolic problems [29,30]. See also [28] for a brief survey in this approach. A similar approach using Cayley transform [12,22] has been reported. We remark that there has been another approach implementing fast inverse Fourier–Laplace transforms using Laguerre polynomials [34,35].

Noble features in our method are as follows. Firstly, it can be easily parallelized, which applies to almost all time evolution problems as mentioned above. Secondly, it reduces significantly the mechanical memories compared to the case that standard time-marching algorithms are applied to the integro-differential equations. As time convolutions are transformed to multiplications by the Fourier–Laplace transformation, the solutions of previous time steps are not necessarily stored.

The frequency domain method is an efficient parallel method for solving time-evolutionary differential equations such as the heat equation, the wave equation, Navier–Stokes equations, visco-elastic equations,
and integro-differential equations. The main character of frequency domain method is using the Fourier–Laplace transformation.

A choice of an adequate contour is a crucial factor in the frequency domain method. The standard Fourier transformation is used in [9, 10, 16, 17], which is equivalent to using the imaginary axis as the contour. Connected two straight lines are used as a contour to solve parabolic problems [29]. A contour which is asymptotically straight lines at infinity is used for an efficient Laplace inversion [31]. The Cayley transform is used to solve first order and convection–diffusion equations in [12, 22], where a parabola is used as a contour. A hyperbola contour is used for parabolic problems in [30]. In this paper, a similar hyperbola contour will be used. Of course, several other contours will give similar results.

The organization of the paper is as follows. In the next section we review known results about the positive memory and the Fourier–Laplace transformation approach. In Section 3 our method of time discretization with the choice of a contour and a transformation integral from the infinite line to a finite interval are described as well as some preliminary estimates. In Section 4, stability and error estimates are given for the method. In Section 5 several numerical examples are considered and analyzed.

2. Positive memory and the frequency domain method

In this paper, the notation $\| \cdot \|$ will be used to denote the $L^2$-norm as well as operator norms wherever no confusion occurs. Also standard notations for Sobolev spaces $H^k(\Omega)$ and the norms $\| \cdot \|_k$ will be used.

2.1. The memory

A memory $K \in L^1([0, \infty))$ is said to be positive-definite if

$$\int_0^T v(t) \int_0^t K(t-s)v(s) \, ds \, dt \geq 0 \quad \text{for all } v \in C([0, T])$$

for all $T > 0$.

Recall first that the Fourier–Laplace transform in time $\hat{v}(z) = \sigma + i\omega = \sigma(\omega) + i\omega$, of a real-valued function $v(t)$ vanishing for $t < 0$ is defined by

$$\hat{v}(z) = \int_0^\infty e^{-zt}v(t) \, dt.$$ \hspace{1cm} (2.1)

If, in particular, $z = i\omega, \omega \in \mathbb{R}$, $\hat{v}$ is the standard one-sided Fourier transform. Plancherel’s theorem implies that $K$ is a positive memory if and only if $\text{Re}(\hat{K}(i\omega)) \geq 0$ for all $\omega \in \mathbb{R}$.

In order to present and analyze our method more concretely, we will assume that $B = A$ in this paper, but several more general cases can be analyzed analogously. We will also restrict ourselves to the following type of memory $K$:

$$K(t) = \frac{e^{-\beta t}t^{x-1}}{\Gamma(x)}$$ \hspace{1cm} (2.2)

for $0 < x < 1$, $\beta \geq 0$, and for $x = 1$, $\beta > 0$; more general type of memories can be treated with more cares. Notice that since $\hat{K}(z) = (z + \beta)^{-x}$ and $\text{Re}(\hat{K}(i\omega)) > 0$ for all $\omega \in \mathbb{R}$, $K$ is a positive memory. Here and in what follows, the branch cut of the argument function log will be chosen as the negative real axis. $K(t) = t^{x-1}, 0 < x < 1$, and $K(t) = e^{-\beta t}, \beta > 0$, are two typical examples of positive memory. For some
technicalities, we treat the case of $0 < \alpha < 1, \beta \geq 0$ in details, while the case $\alpha = 1, \beta > 0$ will be numerically considered in the final section.

2.2. The transformed problem

The Fourier–Laplace transformation of (1.1) with respect to the time variable $t$ yields

$$z\hat{u}(x,z) + (\mu + \hat{K}(z))\hat{A}u(x,z) = u_0(x) + \hat{f}(x,z), \quad (x,z) \in \Omega \times \Sigma,$$

where $\Sigma$ is a region in the complex plane $C$, which will be specified in details later. Formally letting $E(z) = (zI + (\mu + \hat{K}(z))A)^{-1}$, the solution of (2.3) is written as $\hat{u}(z) = E(z)(u_0 + \hat{f}(z))$. If $v$ is square integrable in $t$, that is $\int_0^\infty |v(t)|^2 \, dt < \infty$, the inversion formula

$$\frac{1}{2\pi i} \int_C e^{zt} \hat{v}(z) \, dz = \begin{cases} v(t) & t \geq 0, \\ 0 & t < 0, \end{cases}$$

holds, where $C$ is a contour in the complex plane $C$, and the integration is evaluated as $\text{Im}(z)$ on the contour is increasing from $-\infty$ to $\infty$. If the integral in (2.1) converges for $z_0 = \sigma_0 + io_0$, it converges absolutely and uniformly for all $z$ with $\text{Re}(z) > \text{Re}(z_0)$. The contour $C$ will be chosen to be symmetric with respect to the real axis, and denote by $C_+$ the upper half part of $C$ lying above the real axis. Then, using the conjugacy relation $\hat{v}(z) = \overline{\hat{v}(z)}$, the inversion formula (2.4) takes the simpler form:

$$\frac{1}{\pi} \int_{C_+} e^{zt} \hat{v}(z) \, dz = \begin{cases} v(t) & t \geq 0, \\ 0 & t < 0. \end{cases}$$

With an appropriate contour $C$ contained in $\Sigma$, the solution of (2.3) is computed for $z \in C$ or for $z \in C_+$. Then, the time domain solution $u(t) = u(\cdot, t)$ will be recovered via the following formula:

$$u(x,t) = \frac{1}{2\pi i} \int_C e^{zt} E(z)(u_0(x) + \hat{f}(x,z)) \, dz = \frac{1}{\pi i} \int_{C_+} e^{zt} E(z)(u_0(x) + \hat{f}(x,z)) \, dz.$$

2.3. The time-discretization scheme

We turn to the numerical evaluation of the indefinite integral (2.6). For this, it may be useful to transform $C_+$ into a finite contour by a smooth monotone function $\psi: (-\infty, \infty) \rightarrow [-1, 1]$ such that $\psi(-\omega) = \psi(\omega), \omega \geq 0$ and $\overline{\psi(0, \infty)} = [0, 1]$. In [29] $\psi(\omega) = e^{-\omega t/q}$ is used with parameters $\tau$ and $q$, while $\psi(\omega) = \tanh(\omega^2)$ is used in [30], which shows better behaviors and thus this is again our choice in this paper. To evaluate the resulting integral on the transformed compact contour, we then apply the standard composite trapezoidal rule based on an $N_z$ uniform subdivision of the interval $[0, 1]$. Let $y_j = j/N_z, \, w_j, \, j = 0, \ldots, N_z$, be the quadrature points and weights on $[0, 1]$. Then, for the time-discretization approximation $u_{N_z}(\cdot, t)$ to the solution $u(\cdot, t)$ of (1.1), we have [28]

$$u_{N_z}(\cdot, t) = \frac{1}{\pi} \text{Im} \sum_{j=0}^{N_z} e^{i(\psi^{-1}(y_j)) + i\psi^{-1}(y_j))} \hat{u}(\cdot, \sigma(\psi^{-1}(y_j)) + i\psi^{-1}(y_j)) \{\sigma'(\psi^{-1}(y_j)) + i\} \frac{d\psi^{-1}}{dy}(y_j)w_j$$

(2.7)
since
\[ u(\cdot, t) = \frac{1}{2\pi i} \int_{C} e^{z^2} \hat{u}(\cdot, z) \, dz \]
\[ = \frac{1}{\pi} \Im \int_{C} e^{z^2} \hat{u}(\cdot, z) \, dz \]
\[ = \frac{1}{\pi} \Im \int_{0}^{\infty} e^{(\sigma(\omega) + i\omega)t} \hat{u}(\cdot, \sigma(\omega) + i\omega) \{ \sigma'(\omega) + i \} \, d\omega \]
\[ = \frac{1}{\pi} \Im \int_{0}^{1} e^{(\sigma(\psi^{-1}(y)) + i\psi^{-1}(y))t} \hat{u}(\cdot, \sigma(\psi^{-1}(y)) + i\psi^{-1}(y)) \{ \sigma'(\psi^{-1}(y)) + i \} \frac{d\psi^{-1}}{dy}(y) \, dy. \]

For \( z \in \Gamma \), the solution to problem (2.3) can be approximated by any of the finite element, finite difference, finite volume method, or any other numerical method. Denote by \( \hat{u}_h(z) = \hat{u}_h(\cdot, z) \) the solution to the discrete problem:
\[ z \hat{u}_h(x, z) + (\mu + \hat{K}(z))A_h \hat{u}_h(x, z) = \hat{f}(x, z) + u_0, \quad (x, z) \in \Omega \times \Sigma, \quad (2.8a) \]
\[ \hat{u}_h(x, z) = 0, \quad (x, z) \in \partial \Omega \times \Sigma, \quad (2.8b) \]
where \( A_h \) is the discrete approximation to \( A \). Then the fully-discrete approximation \( u_{h,N}(\cdot, t) \) to the solution of (1.1) can be obtained by (2.7) with \( \hat{u}(\cdot, z) \) in the summand being replaced by \( \hat{u}_h(\cdot, z) \), the solution of (2.8).

**Remark 2.1.** Observe that for each \( j \) the computation of \( \hat{u}_h(\cdot, \sigma(\psi^{-1}(y_j)) + i\psi^{-1}(y_j)) \) satisfying (2.8) is independent of that of \( \hat{u}_h(\cdot, \sigma(\psi^{-1}(y_k)) + i\psi^{-1}(y_k)) \) for \( k \neq j \). This is a crucial fact to our parallel method to be effective since the computation of \( \hat{u}_h(\cdot, \sigma(\psi^{-1}(y_j)) + i\psi^{-1}(y_j)) \) for all \( j \) can be performed in parallel essentially without data communication.

3. The time-discretization scheme

3.1. The contour \( \Gamma \)

There are several ways to choose a suitable contour \( \Gamma \). Being it not essential to have a specific form of the contour to proceed our analysis, it is convenient to work with one concrete example. For this, recalling that \( x \) is given in (2.2), choose an \( \eta \in \left( \frac{1}{2}, \frac{1}{1+2} \right) \), and let
\[ \theta_\eta = (1 - \eta)\pi, \quad 0 < \frac{\pi}{\eta} < \theta_\eta < \frac{\pi}{2}. \]

Let \( \epsilon \) be a sufficiently small positive real number that will be specified in a moment. The analysis in the rest of the paper is then fixed with the \( C^\infty \) contour \( \Gamma \) parameterized by
\[ \Gamma = \{ z | z = z(\omega) = \sigma(\omega) + i\omega, \ \omega \in \mathbb{R} \}, \quad (3.1) \]
where
\[ \sigma(\omega) = \epsilon + \cos \theta_\eta - \sqrt{\omega^2 \cot^2 \theta_\eta + \cos^2 \theta_\eta}, \quad (3.2) \]
with the parameter \( \omega \) running from \(-\infty \) to \( \infty \). In order to emphasize the role of parameters \( \eta \) and \( \epsilon \), we may denote \( \Gamma = \Gamma_{\eta, \epsilon} \). The properties of \( \sigma(\omega) \) that are useful in our analysis are listed as follows:
\[ \sigma(\omega) \leq \epsilon + \cos \theta_\eta - |\omega| \cot \theta_\eta \quad \text{for all } \omega \in \mathbb{R}, \quad (3.3a) \]
\[ \sigma(\omega) = \epsilon + \cos \theta_\eta - |\omega| \cot \theta_\eta + O(|\omega|^{-1}) \quad \text{as } |\omega| \to \infty, \quad (3.3b) \]

\[ \left| \frac{d\sigma(\omega)}{d\omega} \right| = O(1); \quad \left| \frac{d^k\sigma(\omega)}{d\omega^k} \right| = O(|\omega|^{-k-1}) \quad \text{for } k \geq 2, \quad \text{as } |\omega| \to \infty, \quad (3.3c) \]

Let \( \gamma(r) := \max_{1 \leq k \leq r} \sup_{z \in \mathbb{R}} \left| \frac{d^k\sigma(\omega)}{d\omega^k} \right| \), then \( \gamma(r) < \infty \) for each integer \( r \geq 1 \). \quad (3.3d)

Let \( \Sigma \) be a sufficiently thin neighborhood of \( \Gamma_{\eta, \epsilon} \) such that the negative real axis containing the origin 0 is to the left of \( \Sigma \). Since the spectrum \( \text{Spec}(A) \) of \( A \) lies on the positive real axis, \( zI + (\mu + \hat{K}(z))A \) is invertible and thus its inverse \( E(z) \) is well defined for all \( z \) in \( \Sigma \). Moreover \( \hat{K} \) is \( C^\infty \) in \( \Sigma \).

Due to (3.3), in this subsection, for later use, we will bound \( \| \frac{d^k E(z)}{dz^k} \|, k \geq 0 \) and \( |z + \beta|^\gamma \), \( \gamma \geq 1 \).

Choose a sufficiently small \( \epsilon' \) such that \( B_{\epsilon'}(0) \) lies to the left of \( \Gamma_{\eta, \epsilon} \). Then

\[ |z + \beta| \geq |\omega| \geq \frac{1}{2}(1 + |\omega|) \quad \text{for } |\omega| > 1, \quad (3.4a) \]

\[ |z + \beta| \geq \epsilon' \geq \epsilon'(1 + |\omega|) \quad \text{for } |\omega| \leq 1, \quad (3.4b) \]

\[ |z + \beta| \leq |z| + \beta \leq \left(1 + \frac{\beta}{\epsilon'}\right)|z|. \quad (3.4c) \]

By (3.4a) and (3.4b), one has

\[ |z + \beta|^\gamma \geq C(1 + |\omega|)^\gamma \geq C(1 + |\omega|), \quad (3.5) \]

where \( \gamma \) is a positive real number greater than 1 and \( C \) is a constant depending on \( \epsilon' \) and \( \gamma \).

It is convenient to write

\[ zI + (\mu + \hat{K}(z))A = (\mu + \hat{K}(z)) \left( \frac{z}{\mu + \hat{K}(z)}I + A \right) \]

in order to bound the derivatives of the solution operator \( E(z) \) with respect to \( z \). For \( 0 < \eta' < 1 \), set

\[ \Sigma_{\eta'} = \{ z \in \mathbb{C} || \arg(z) | < \eta' \pi \}. \]

Next let \( \delta := (1 + \epsilon)\eta \). Then \( \frac{1}{2} < \frac{1 + \epsilon}{2} < \delta < 1 \) and

\[ \frac{z}{\mu + \hat{K}(z)} \in \Sigma_\delta \quad \text{if } z \in \Sigma_\eta. \quad (3.6) \]

If \( z \in \Sigma_\eta, \frac{z}{\mu + \hat{K}(z)} \in \Sigma_\delta \) and \( \frac{\hat{K}(z)}{z} \in \Sigma_\delta \). Using the fact that \( z_1, z_2 \in \Sigma_\delta \) implies \( z_1 + z_2 \in \Sigma_\delta \), we can show that \( \frac{\mu + \hat{K}(z)}{z} \in \Sigma_\delta \) which is equivalent to (3.6).

For \( \lambda \in \text{Spec}(A) \), by the assumption (3.6) and the fact that \( \frac{1}{2} < \frac{1 + \epsilon}{2} < \delta < 1 \), a simple computation implies that

\[ |\lambda + \zeta| \geq |\zeta| \sin \delta \pi, \quad (3.7a) \]

\[ |\lambda + \zeta| \geq \lambda \sin \delta \pi \quad (3.7b) \]
if $|\arg(\zeta)| < \delta\pi$. From (3.7a) it follows that $E(z)$ is bounded in the operator norm as follows:

$$\|E(z)\| \leq \frac{1}{|\mu + \tilde{K}(z)|} \inf_{\lambda \in \text{Spec}(\tilde{K})} \left| \frac{1}{\lambda + \frac{z}{\mu z(\lambda)}} \right| \leq \frac{1}{\sin \delta\pi |z|}$$

(3.8)

for $z \in \Sigma_n$. Using $\tilde{K}(z) \in \Sigma_{2n}$ and a similar analysis as (3.7a), we get the lower bound of $|\mu + \tilde{K}(z)|$:

$$|\mu + \tilde{K}(z)| \geq |\tilde{K}(z)| \sin 2\eta\pi.$$  

(3.9)

With the contour $\Gamma_{n,r}$, proceed further to the convergence analysis of our method. Due to (3.1) and (3.2), the Eq. (2.6) can be rewritten as

$$u(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} v(\omega) d\omega, \quad v(\omega) = \hat{u}(\omega) \frac{1}{t} \frac{d\hat{e}^{i\omega t}}{d\omega} = \hat{u}(\omega) e^{i\omega t} x^t(\omega).$$

(3.10)

3.2. Transformation of integral from the infinite line to a finite interval

The integration in (3.10) over the infinite interval will be computed after it is transformed into that over a finite interval by a transformation $\psi : (-\infty, \infty) \to [-1, 1]$ given by $y = \psi(\omega) = \tanh(\frac{\pi\omega}{2})$, as introduced in the beginning of Section 2.3. For this, write

$$u(t) = \frac{1}{2\pi i} \int_{-1}^{1} V(y) dy, \quad V(y) = v(\psi^{-1}(y)) \frac{d\psi^{-1}(y)}{dy} = \frac{v(\omega)}{\psi'(\omega)}.$$  

(3.11)

Then the composite trapezoidal rule with the $2N_z$ uniform subdivision of $[-1, 1]$ is applied for the integration of (3.11) as follows:

$$U_{N_z}(t) = \frac{1}{2\pi i} \frac{1}{N_z} \sum_{j=-N_z+1}^{N_z-1} V(y_j) = \frac{1}{2\pi i} \frac{1}{N_z} \sum_{j=-N_z+1}^{N_z-1} v(\omega_j)\mu_j, \quad V(y) = \frac{v(\omega)}{\psi'(\omega)},$$

(3.12)

where

$$y_j = \frac{j}{N_z}, \quad \omega_j = \psi^{-1}(y_j) = \frac{1}{\tau} \log \frac{1 + y_j}{1 - y_j}, \quad \frac{1}{\psi'(y_j)} = \frac{2}{\tau} \frac{1}{1 - y_j^2}, \quad \mu_j = \frac{2}{1 - y_j^2}.$$  

Notice that (3.12) is of arbitrarily high order of accuracy, provided that $v(\omega)$ vanishes fast at infinity, due to the following Euler–Maclaurin type lemma:

**Lemma 3.1.** [30] Let $r \geq 1$ be given and assume that $v \in C^r(\mathbb{R}; L^2(\Omega))$ and that

$$\left| \frac{d^j v}{d\omega^j}(\omega) \right| = O(e^{-\tau|\omega|}) \quad \text{as} \quad |\omega| \to \infty \quad \text{for} \quad 0 \leq j \leq r,$$

(3.13)

and if $r = 1$ also that $\|v(\omega)\| = O(e^{-\tau|\omega|})$, where $\tau$ is the parameter in (3.12). Then the following estimate holds:

$$\left\| U_{N_z}(t) - \frac{1}{2\pi i} \int_{-\infty}^{\infty} v(\omega) d\omega \right\| \leq C_\tau \tau^{-r} \left( 1 + \tau^{-r} \right) \int_{-\infty}^{\infty} e^{\tau|\omega|} \sum_{j=0}^{r} \left\| \frac{d^j v}{d\omega^j} \right\| d\omega.$$  

(3.14)

4. Stability and error estimates

We will first analyze stability of our quadrature scheme and then obtain an error estimate using Lemma 3.1.
4.1. The stability estimate

We may rewrite (3.12) in the following form:

$$U_{N_e}(t) = \frac{1}{2\pi i} \sum_{j=-N_e+1}^{N_e-1} \frac{1}{\tau N_e} \bar{\mu}_j e^{\gamma j \tau} \hat{u}(z_j), \quad \bar{\mu}_j = z'(\omega_j)\mu_j, \quad z_j = z(\omega_j) = \sigma(\omega_j) + i\omega_j. \quad (4.1)$$

Our stability estimate is then given in the following form:

**Theorem 4.1.** Let $U_{N_e}(t)$ be the approximate solution of (1.1) given by (4.1). Assume that $\hat{f}(z)$ is analytic in $z$ to the right of the contour $\Gamma_{\eta,c}$ and continuous up to $\Gamma_{\eta,c}$. Then

$$\|U_{N_e}(t)\| \leq Ce^{\gamma_0 + \log_+ \left( 1 + \frac{1}{\tau N_e} + \log_+ \frac{\tan \theta_j}{t} \right) \left( \|u_0\| + \sup_{z \in \Gamma_{\eta,c}} \|\hat{f}(z)\| \right)}, \quad t > 0. \quad (4.2)$$

**Proof.** Since $|\bar{\mu}_j| \leq (\gamma(1) + 1)\mu_j$, by (3.3a), (3.3d), and (4.1), we have

$$\|U_{N_e}(t)\| \leq \frac{1}{2\pi} \frac{1}{\tau N_e} (\gamma(1) + 1) e^{(\gamma + \cot \theta_0)\tau} \sum_{j=-N_e+1}^{N_e-1} \mu_j e^{-\cot \theta_0 |\omega_j|t} \|\hat{u}(z_j)\|. \quad (4.3)$$

But by (3.8) we have the bound for $\|\hat{u}(z_j)\|$

$$\|\hat{u}(z_j)\| = \|E(z_j)(u_0 + \hat{f}(z_j))\| \leq \frac{C}{|\omega_j|} (\|u_0\| + \|\hat{f}(z)\|) \quad (4.4)$$

and therefore we need to bound the term

$$\frac{C}{\tau N_e} \sum_{j=-N_e+1}^{N_e-1} \mu_j e^{-\cot \theta_0 |\omega_j|t} \frac{1}{|\omega_j|}. \quad (4.5)$$

Using (3.5), (4.4) is bounded by

$$\frac{C}{\tau N_e} \sum_{j=-N_e+1}^{N_e-1} \mu_j e^{-\cot \theta_0 |\omega_j|t} \frac{1}{1 + |\omega_j|}. \quad (4.5)$$

The term with $j = 0$ is bounded by $\frac{C}{\tau N_e}$. For $j > 0$, since $e^{-\cot \theta_0 \psi^{-1}(y) t} / (1 + \psi^{-1}(y))$ is decreasing in $y$ for $y \geq 0$, by using $y_j = j/N_e$ and $\frac{dy}{dy} = (1 - y^2) \tau / 2$,

$$\frac{e^{-\cot \theta_0 \psi^{-1}(y) t} / (1 + \psi^{-1}(y))}{1 + \omega_j} \leq N_e \int_{y_{j-1}}^{y_j} e^{-\cot \theta_0 \psi^{-1}(y) t} \frac{dy}{1 + \psi^{-1}(y)} = \frac{\tau N_e}{2} \int_{\omega_{j-1}}^{\omega_j} e^{-\cot \theta_0 \omega} (1 - \psi(\omega)^2) \, d\omega.$$

Hence by using $\mu_j = 2 / (1 - y_j^2)$ and the easily shown fact that $1 - y^2 \leq 2(1 - y_j^2)$ for $y_{j-1} \leq y \leq y_j$ for $1 \leq j \leq N_e - 1$, we obtain

$$\frac{1}{\tau N_e} \sum_{j=1}^{N_e-1} \mu_j \frac{e^{-\cot \theta_0 \omega_j t}}{1 + \omega_j} \leq 2 \int_{0}^{\infty} \frac{e^{-\cot \theta_0 \omega}}{1 + \omega} \, d\omega \leq C \left( 1 + \log_+ \left( \frac{\tan \theta_j}{t} \right) \right), \quad (4.5)$$

where $\log_+ x = \max(0, \log x)$. The argument for $j < 0$ is analogous. Thus, by combining (4.3)–(4.5) for all $j$, the theorem is obtained.
4.2. The error analysis

In order to apply Lemma 3.1 for our error analysis, we need to estimate the term $\sum_{j=0}^{r} ||d^j E/\partial \omega||$ in the lemma. For this we begin with investigating the derivatives of $E(z)$ with respect to $z$. Recalling the definition of $E(z)$ and the Leibniz differentiation rule of product of two functions, one can verify that

$$\frac{d^j E(z)}{dz^j} = -E(z) \sum_{k=0}^{j-1} \left( j \atop k \right) \frac{d^{j-k}(z^k + (\mu + \hat{K}(z))A}{dz^{j-k}} \frac{d^j E(z)}{dz^j}. \quad (4.6)$$

Therefore, using (3.4c), (3.7b), (3.8) and (3.9), we get

$$\left\| E(z) \frac{d^k(\mu + \hat{K}(z))}{dz^k} A \right\| \leq \frac{1}{|\mu + \hat{K}(z)|} \inf_{\lambda \in \text{Spec}(A)} \left| \lambda + \frac{z}{\mu + \hat{K}(z)} \right| \leq \frac{1}{\sin z \eta \pi \sin \delta \pi} \frac{1}{|z + \beta|} \quad \text{for } k \geq 1,$$

(4.7a)

$$\left\| E(z) \left( I + \frac{d(\mu + \hat{K}(z))}{dz} A \right) \right\| \leq \|E(z)\| + \left\| E(z) \frac{d(\mu + \hat{K}(z))}{dz} A \right\| \leq \left[ \frac{1 + \beta}{\sin \delta \pi} + \frac{1}{\sin z \eta \pi \sin \delta \pi} \right] \frac{1}{|z + \beta|} \quad (4.7b)$$

for $z \in \Gamma_{\eta, \epsilon}$. By using (4.6), (4.7), and an induction on $j$, an upper bound of the derivatives of $E(z)$ with respect to $z$ can be given as follows:

$$\left\| \frac{d^j E(z)}{dz^j} \right\| \leq C(\epsilon, \beta, \eta, \epsilon, j) \frac{1}{|z + \beta|^j} \quad (4.8)$$

for $z \in \Gamma_{\eta, \epsilon}$, since $\delta = (1 + z)\eta$ and $e'$ is defined by $\epsilon$ and $\eta$. Since $\bar{u}(z) = E(z)(u_0 + \hat{f}(z))$, by utilizing

$$\frac{d^j \bar{u}(z)}{dz^j} = \sum_{k=0}^{j} \left( j \atop k \right) \frac{d^k E(z)}{dz^k} \frac{d^{j-k}(u_0 + \hat{f}(z))}{dz^{j-k}}$$

and (4.8), an upper bound of the derivatives of $\bar{u}(z)$ with respect to $z$ is obtained as follows:

$$\left\| \frac{d^j \bar{u}(z)}{dz^j} \right\| \leq C(\epsilon, \beta, \eta, \epsilon, j) \left( \frac{1}{|z + \beta|} + \frac{1}{|z + \beta|^j} \right) \left( \|u_0\| + \max_{0 \leq k < j} \sup_{z \in \Gamma_{\eta, \epsilon}} \left\| \frac{d^k \hat{f}(z)}{dz^k} \right\| \right) \quad (4.9)$$

for $z \in \Gamma_{\eta, \epsilon}$.

Next, the derivatives of $\bar{u}(z(\omega))$ and $e^{z(\omega)t}$ with respect to $\omega$ are bounded, as in [30], by induction as follows:

$$\frac{d^j \bar{u}(z(\omega))}{d\omega^j} = \sum_{k=1}^{j} \frac{d^k \bar{u}}{dz^k} Q_{j,k} \left( \frac{dz}{d\omega}, \frac{d^2z}{d\omega^2}, \frac{d^3z}{d\omega^3}, \ldots, \frac{d^jz}{d\omega^j} \right), \quad (4.10a)$$

$$\frac{d^j e^{z(\omega)t}}{d\omega^j} = \sum_{k=1}^{j} \frac{d^k e^{z(\omega)t}}{dz^k} Q_{j,k} \left( \frac{dz}{d\omega}, \frac{d^2z}{d\omega^2}, \frac{d^3z}{d\omega^3}, \ldots, \frac{d^jz}{d\omega^j} \right), \quad (4.10b)$$
where \( j \geq 1 \) and \( Q_j^k(\omega_1, \omega_2, \ldots, \omega_j) \) is a linear combination of monomials \( \prod_{i=1}^j \omega_i^{l_i} \) with \( \sum_{i=1}^j l_i = j \), \( 0 \leq j \leq j \). Due to (3.3), we have

\[
\left| Q_j^k \left( \frac{dz}{d\omega}, \frac{d^2z}{d\omega^2}, \frac{d^3z}{d\omega^3}, \ldots, \frac{d^jz}{d\omega^j} \right) \right| \leq C\gamma(j),
\]

where \( \gamma(j) \) is a constant defined by the contour \( \Gamma_{\eta, \varepsilon} \), thus a constant depending on \( \eta, \varepsilon, j \). For \( j \geq 1 \), upper bounds of \( j \)th derivatives of \( \tilde{u}(\varphi(\omega)) \) and \( e^{\varphi(\omega)} \) are given combining (4.9), (4.10), and (4.11)

\[
\left\| \frac{d^j\tilde{u}(\varphi(\omega))}{d\omega^j} \right\| \leq C \left( \left\| u_0 \right\| + \max_{0 \leq k \leq j} \sup_{z \in \Gamma_{\eta, \varepsilon}} \left\| \frac{d^k\tilde{f}(z)}{dz^k} \right\| \right),
\]

(4.12a)

\[
\left\| \frac{d^j e^{\varphi(\omega)}}{d\omega^j} \right\| \leq (t + t') e^{\varphi(\omega)},
\]

(4.12b)

with the constant \( C \) dependent on \( x, \beta, \eta, \varepsilon, j \). Finally, let us consider the derivatives of \( v \) with respect to \( \omega \):

\[
\frac{dv}{d\omega}(\omega) = \sum_{k=0}^j \binom{j}{k} \frac{d^k\tilde{u}(\varphi(\omega))}{d\omega^k} \frac{1}{1+|\omega|} \left( \left\| u_0 \right\| + \max_{0 \leq k \leq j} \sup_{z \in \Gamma_{\eta, \varepsilon}} \left\| \frac{d^k\tilde{f}(z)}{dz^k} \right\| \right),
\]

Using the estimates (4.12) and (3.5), we have an upper bound of \( v \) with respect to \( \omega \) as follows:

\[
\left\| \frac{d^j v(\omega)}{d\omega^j} \right\| \leq C(t + t') e^{\varphi(\omega)} \frac{1}{1+|\omega|} \left( \left\| u_0 \right\| + \max_{0 \leq k \leq j} \sup_{z \in \Gamma_{\eta, \varepsilon}} \left\| \frac{d^k\tilde{f}(z)}{dz^k} \right\| \right),
\]

where the coefficient \( C \) depends on \( x, \beta, \eta, \varepsilon, j \).

Now we are in a position to state and prove one of our main results:

**Theorem 4.2.** Let \( \mu \) be a nonnegative real number and \( K(t) = e^{-\beta t} t^{r-1} / \Gamma(x) \) be a positive memory where \( 0 < x < 1 \) and \( \beta \geq 0 \). Choose \( \eta \in (\frac{1}{2}, \frac{1}{r-1}) \) and \( \epsilon > 0 \). Then we can choose the region \( \Sigma_\eta \) in the complex domain and the contour \( \Gamma_{\eta, r} \) as in (3.1) contained in the region \( \Sigma_\eta \) satisfying (3.3). Assume that \( \tilde{f}(z) \) is analytic to the right of the contour \( \Gamma_{\eta, r} \) and continuous up to \( \Gamma_{\eta, r} \), and that \( |\tilde{f}^{(r)}(z)| \) is uniformly bounded on the contour \( \Gamma_{\eta, r} \) for \( 0 \leq j \leq r \) with \( r \geq 2 \). Let \( u(t) \) be the solution of (1.1) and \( U_{\eta, r} \) the approximated solution in (3.12). With \( p = rt \tan \theta_0 \), we have for \( t > rt \tan \theta_0 \), (equivalently, \( 0 < p < 1 \)),

\[
\| U_{\eta, r}(t) - u(t) \| \leq CN_t(1 + t^{-r}) e^{(r+\cos \theta_0)} \left( 1 + \log_+ \left( \frac{\tan \theta_0}{t(1-p)} \right) \right) \left( \| u_0 \| + \max_{0 \leq k \leq r} \sup_{z \in \Gamma_{\eta, r}} \left\| \frac{d^k\tilde{f}(z)}{dz^k} \right\| \right),
\]

(4.14)

with \( C = C(T, x, \beta, \eta, \epsilon, r, p) > 0 \).

**Proof.** By the estimate (4.13) and the condition \( t > rt \tan \theta_0 \) with the behavior of \( \Gamma_{\eta, x} \) at infinity appeared in (3.3a), (3.3b), (3.13) is satisfied. Thus by Lemma 3.1, we get (3.14). To prove the theorem it is sufficient to consider the following inequality

\[
\int_{-\infty}^{\infty} e^{\varphi(\omega)} \frac{d\omega}{1 + |\omega|} \leq e^{\varphi(\omega)} \int_{-\infty}^{\infty} \frac{e^{(r+\cos \theta_0)} \cot \theta_0}{1 + |\omega|} \frac{d\omega}{1 + |\omega|} \leq C e^{(r+\cos \theta_0)} \left( 1 + \log_+ \left( \frac{\tan \theta_0}{t(1-p)} \right) \right),
\]

where the last inequality is derived using similar analysis used in (4.5). \( \square \)
Remark 4.3. We remark that (4.14) holds for \( t > r\tan \theta_k \), where \( \theta_k \) is determined by the kernel \( K(t) \) and the contour while \( r \) is the regularity of the forcing term. Therefore, if one seeks a reasonable approximate solution at \( t_0 \), one is recommended to choose \( r \) satisfying \( r < \frac{t_0}{\tan \theta_k} \).

4.3. Space discretization and comparison to time-stepping methods

First, let us consider time-discretization error. Let \( S_h^k \) be \((k-1)\)th degree finite element space with largest mesh size \( h \) and \( R_h \) be Ritz projection. Consider the finite element approximation \( u_h(t) \in S_h^k \) satisfying

\[
(u_h, v) + \mu A(u_h, v) + \int_0^t K(t-s)A(u_h(s), v)\,ds = (f, v), \quad v \in S_h^k,
\]

(4.15a)

\[
u_h(0) = u_{0h} = R_h u_0.
\]

(4.15b)

If \( K \) is a positive memory, it is well known that

\[
||u_h - u||_l \leq C \left(||u_{0h} - u_0||_l + h^k \left(||u_0||_{l_k} + \int_0^t ||u_i||_{l_k}\,ds\right)\right),
\]

(4.16a)

\[
||u_h - u||_1 \leq C \left(||u_{0h} - u_0||_1 + h^{k-1} \left(||u_0||_{l_k} + \int_0^t ||u_i||_{l_k}\,ds\right)\right).
\]

(4.16b)

Notice that the full-discretization error of the frequency domain method will be dominated by the sum of (4.14) and (4.16), that is, \( O(h^2 + N_x^{-r}) \) (see also [30]). This fact is illuminating by comparing this bound to the orders of convergence of time stepping methods, for instance, the backward Euler and Crank–Nicolson methods which are given as \( O(h^2 + N_t^{-1}) \) and \( O(h^2 + N_t^{-(1+\mu)}) \), respectively for \( \mu = 0 \) (see [3]). Here, \( N_t \) is the number of uniform subdivisions in the time interval.

Suppose that the space mesh size \( h \) is equal to \( 1/N_x \). As for time stepping methods, the optimal relation between \( N_t \) and \( N_x \) is \( N_t = CN_x^{1/2} \), \( N_t = CN_x^{|\mu|} \) for backward Euler and Crank–Nicolson methods, respectively, which result in \( O(N_x^{-2}) \) convergence on both case. This is implemented in Tables 4 and 8 of [3]. Whereas in the frequency domain method, the optimal relation between \( N_t \) and \( N_x \) is \( N_t = CN_x^{1/2} \) resulting in \( O(N_x^{-r}) \), \( r \geq 2 \) convergence. Thus the frequency domain method is a much faster method depending on parameter \( r \), which is related with the regularity of \( \hat{f} \), than usual time stepping methods. Various strategies of balancing the frequency and space steps in the frequency domain method are compared to the time stepping methods: in these cases, convergence rates and the size of linear matrix system are described in Table 1.

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comparison of the methods with ( r = 2, 3, ) and 4 with the time-stepping BE (backward Euler) and CN (Crank–Nicolson) methods in ( d ) dimension, where ( \mu = 0 ), ( K(t) = \frac{\nu^2}{r\tau} ).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Convergence</th>
<th>Size of system matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>BE</td>
<td>( N_t = CN_x^{1/2} )</td>
<td>( O(N_x^{-2}) )</td>
</tr>
<tr>
<td>CN</td>
<td>( N_t = CN_x^{1/2} )</td>
<td>( O(N_x^{-2}) )</td>
</tr>
<tr>
<td>( r = 2 )</td>
<td>( \mu = 0 )</td>
<td>( O(N_x^{-2}) )</td>
</tr>
<tr>
<td>( r = 3 )</td>
<td>( \mu = 1/2 )</td>
<td>( O(N_x^{-3}) )</td>
</tr>
<tr>
<td>( r = 4 )</td>
<td>( \mu = 2/2 )</td>
<td>( O(N_x^{-4}) )</td>
</tr>
</tbody>
</table>
In this section numerical examples are presented in order to confirm the theory developed in the previous sections and to have ideas for future studies. The following six example kernels are treated:

Case 1  \[ K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha = 0.5, \quad \mu = 0, \]  
(5.1a)

Case 2  \[ K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha = 0.5, \quad \mu = 1, \]  
(5.1b)

Case 3  \[ K(t) = \frac{e^{-\beta t^{\alpha-1}}}{\Gamma(\alpha)}, \quad \alpha = 0.5, \quad \beta = 1, \quad \mu = 0, \]  
(5.1c)

Case 4  \[ K(t) = \frac{e^{-\beta t^{\alpha-1}}}{\Gamma(\alpha)}, \quad \alpha = 0.5, \quad \beta = 1, \quad \mu = 1, \]  
(5.1d)

Case 5  \[ K(t) = e^{-\beta t}, \quad \beta = 1, \quad \mu = 0, \]  
(5.1e)

Case 6  \[ K(t) = e^{-\beta t}, \quad \beta = 1, \quad \mu = 1. \]  
(5.1f)

In Cases 5 and 6, where \( \alpha = 1 \), the contour in (3.1) is not defined since \( \alpha \pi(1 + \alpha) = \pi/2 \), and thus in order to define our contour in a deformed curve as described in Section 2 an approximate value \( \alpha = 0.9 \) is used in the numerical implementation preserving \( K(t) \) as in (5.1c). Alternatively the \( y \)-axis can be used as a contour \([9,10,16,17]\). The choice of parameters \( \delta = 0.9, \epsilon = 0.1, \) and \( \eta = \frac{\epsilon}{\delta} \) defines the contour \( \Gamma_{\eta,\epsilon} \) uniquely and is used in also our numerical integration (3.12). Thus the same contour for Cases 1, 2, 3, and 4 is chosen and also for Cases 5 and 6.

To verify (4.14), let us consider a zero-dimensional case in order to investigate the time-discretization error only with the application of the frequency domain method neglecting the space-discretization error completely. As suggested in Remark 4.3, if we would like to have the solution at time \( t = 1 \) with the order \( r = 2 \), we may take \( \tau < \frac{1}{\tan \eta} \). In our numerical examples, we simply choose \( \tau = \frac{1}{\tan \eta} \). Thus if we use the same \( \tau \) for all time \( t = 0.2, 0.4, \ldots, 2 \), which will be investigated in the following analysis, the convergence order will be changed according to the time \( t \). In other words, the convergence order \( r \) may depend on the time \( t \) as \( r < r_{\max} := \frac{1}{\tan \eta} t, \) We specifically consider the problem to find a real-valued function \( u : \mathbb{R}^+ \to \mathbb{R} \) satisfying

\[
  u' + \int_0^t K(t-s)u(s)\,ds = f(t) \quad \text{in } [0, T],
\]

(5.2a)

\[
  u(0) = u_0.
\]

(5.2b)

If we choose the initial data and the forcing term such as \( u_0 = 1 \) and \( f(t) = -e^{-t} + \mu e^{-t} + \int_0^t K(t-s)e^{-s}\,ds \), then the unique solution of (1.1) is \( u(t) = e^{-t} \). Let \( e_{N_c} \) be the \( L^2 \) error when we extract \( N_c \) points on the contour in the frequency domain, and denote by \( \rho_{N_c} \) the convergence order \( \log_2(e_{N_c}/e_{N_c'}) \). Tables 2 and 3 show apparent orders of convergence for Cases 1 and 5, and suggest us how many points to choose on the contour in order to reach accuracy needed in actual computations for spatially two or three-dimensional problems. In these tables, \( \frac{\rho_{N_c} + \rho_{N_c'}}{2} \) denotes \( \log_{128/32}(e_{32}/e_{128}) \) meaning the average convergence order from \( N_c = 32 \) to \( N_c = 128 \). These values increase as time passes in Tables 2 and 3. Especially, in Case 1, they are
approximately 2. The errors in Case 5 are less than those in Case 1. This is due to the fact that we use an approximation of \( z \) in Case 5, which will be also employed in the one- and two-dimensional cases to follow.

Let us proceed to one-dimensional cases. To consider space discretization besides frequency discretization, let us consider the strategy given in Table 1 such as

### Table 2: Errors and apparent convergence rates for the frequency-discretization algorithm applied to the zero-dimensional example with the choice of \( N_x = 16, 32, 64, 128, \epsilon = 0.1, \delta = 0.9 \) and \( \tau = \frac{1}{2 \tan \nu_0} \approx 0.289 \)

<table>
<thead>
<tr>
<th>( N_x )</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>( \rho_{128} )</th>
<th>(( \rho_{64} + \rho_{128} ))/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.122E-01</td>
<td>0.117E-01</td>
<td>0.052</td>
<td>0.854E-02</td>
<td>0.460</td>
<td>0.492E-02</td>
</tr>
<tr>
<td>0.4</td>
<td>0.151E-02</td>
<td>0.133E-02</td>
<td>0.186</td>
<td>0.399E-03</td>
<td>1.736</td>
<td>0.106E-03</td>
</tr>
<tr>
<td>0.6</td>
<td>0.261E-03</td>
<td>0.196E-03</td>
<td>0.409</td>
<td>0.146E-04</td>
<td>3.749</td>
<td>0.300E-04</td>
</tr>
<tr>
<td>0.8</td>
<td>0.440E-04</td>
<td>0.337E-04</td>
<td>0.385</td>
<td>0.956E-05</td>
<td>1.819</td>
<td>0.931E-06</td>
</tr>
<tr>
<td>1.0</td>
<td>0.125E-05</td>
<td>0.658E-05</td>
<td>2.399</td>
<td>0.206E-05</td>
<td>1.680</td>
<td>0.348E-06</td>
</tr>
<tr>
<td>1.2</td>
<td>0.595E-05</td>
<td>0.148E-05</td>
<td>2.007</td>
<td>0.316E-06</td>
<td>2.231</td>
<td>0.610E-07</td>
</tr>
<tr>
<td>1.4</td>
<td>0.418E-05</td>
<td>0.394E-06</td>
<td>3.405</td>
<td>0.312E-07</td>
<td>3.660</td>
<td>0.177E-08</td>
</tr>
<tr>
<td>1.6</td>
<td>0.321E-05</td>
<td>0.125E-06</td>
<td>4.684</td>
<td>0.143E-08</td>
<td>6.452</td>
<td>0.123E-08</td>
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<tr>
<td>1.8</td>
<td>0.222E-06</td>
<td>0.452E-07</td>
<td>2.294</td>
<td>0.196E-08</td>
<td>4.524</td>
<td>0.314E-09</td>
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<tr>
<td>2.0</td>
<td>0.147E-05</td>
<td>0.176E-07</td>
<td>6.387</td>
<td>0.758E-09</td>
<td>4.538</td>
<td>0.387E-10</td>
</tr>
</tbody>
</table>

### Table 3: Errors and apparent convergence rates for the frequency-discretization algorithm applied to the zero-dimensional example with the choice of \( N_x = 16, 32, 64, 128, r = 2, \epsilon = 0.1, \delta = 0.9 \) and \( \tau = \frac{1}{2 \tan \nu_0} = 0.041 \)

<table>
<thead>
<tr>
<th>( N_x )</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>( \rho_{128} )</th>
<th>(( \rho_{64} + \rho_{128} ))/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.926E-01</td>
<td>0.214E-01</td>
<td>2.115</td>
<td>0.665E-02</td>
<td>1.683</td>
<td>0.421E-02</td>
</tr>
<tr>
<td>0.4</td>
<td>0.102E+00</td>
<td>0.127E-01</td>
<td>3.007</td>
<td>0.908E-03</td>
<td>3.808</td>
<td>0.150E-03</td>
</tr>
<tr>
<td>0.6</td>
<td>0.795E-01</td>
<td>0.696E-02</td>
<td>3.515</td>
<td>0.803E-04</td>
<td>6.438</td>
<td>0.953E-04</td>
</tr>
<tr>
<td>0.8</td>
<td>0.662E-01</td>
<td>0.633E-02</td>
<td>3.387</td>
<td>0.614E-04</td>
<td>6.687</td>
<td>0.407E-04</td>
</tr>
<tr>
<td>1.0</td>
<td>0.548E-01</td>
<td>0.510E-02</td>
<td>3.426</td>
<td>0.101E-03</td>
<td>5.665</td>
<td>0.240E-05</td>
</tr>
<tr>
<td>1.2</td>
<td>0.518E-01</td>
<td>0.443E-02</td>
<td>3.547</td>
<td>0.387E-04</td>
<td>6.838</td>
<td>0.298E-05</td>
</tr>
<tr>
<td>1.4</td>
<td>0.573E-01</td>
<td>0.413E-02</td>
<td>3.794</td>
<td>0.506E-04</td>
<td>6.350</td>
<td>0.828E-06</td>
</tr>
<tr>
<td>1.6</td>
<td>0.702E-01</td>
<td>0.382E-02</td>
<td>4.198</td>
<td>0.342E-04</td>
<td>6.806</td>
<td>0.136E-06</td>
</tr>
<tr>
<td>1.8</td>
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<td>0.377E-02</td>
<td>4.788</td>
<td>0.335E-04</td>
<td>6.814</td>
<td>0.952E-07</td>
</tr>
<tr>
<td>2.0</td>
<td>0.331E+00</td>
<td>0.417E-02</td>
<td>6.309</td>
<td>0.348E-04</td>
<td>6.906</td>
<td>0.516E-08</td>
</tr>
</tbody>
</table>

Let us turn to two-dimensional problems. Let \( \Omega = [0, \pi] \times [0, \pi], A = -\frac{\partial^2}{\partial x^2}, \) and initial data and the forcing terms be chosen such that \( u_0(x, y) = \sin x \sin y \) and \( f(x, t) = \left( (1 \pm \mu) e^{-t} + \int_0^t K(t - s)e^{-s} ds \right) \sin x \sin y \) with the unique solution of (1.1) being \( u(x, t) = e^{-t} \sin x \sin y. \) For each frequency, the complex-valued
Table 4
Case 2: Errors and apparent convergence rates for the fully-discretization algorithm applied to the one-dimensional example with the choice of $N_x = 16, 32, 64, N_z = \frac{\pi}{2.14}, \epsilon = 0.1, \delta = 0.9$ and $\tau = \frac{1}{2\sqrt{N_x}} \approx 0.144$

<table>
<thead>
<tr>
<th>$N_z(N_x)$</th>
<th>16(16)</th>
<th>32(64)</th>
<th>$\rho_{12}$</th>
<th>64(256)</th>
<th>$\rho_{64}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.593E-03</td>
<td>0.167E-03</td>
<td>1.831</td>
<td>0.259E-04</td>
<td>2.684</td>
</tr>
<tr>
<td>0.4</td>
<td>0.207E-03</td>
<td>0.467E-05</td>
<td>5.470</td>
<td>0.519E-06</td>
<td>3.170</td>
</tr>
<tr>
<td>0.6</td>
<td>0.159E-03</td>
<td>0.108E-05</td>
<td>7.205</td>
<td>0.161E-07</td>
<td>6.067</td>
</tr>
<tr>
<td>0.8</td>
<td>0.120E-03</td>
<td>0.975E-06</td>
<td>6.948</td>
<td>0.306E-07</td>
<td>4.992</td>
</tr>
<tr>
<td>1.0</td>
<td>0.780E-04</td>
<td>0.941E-06</td>
<td>6.374</td>
<td>0.303E-07</td>
<td>4.958</td>
</tr>
<tr>
<td>1.2</td>
<td>0.306E-04</td>
<td>0.871E-06</td>
<td>5.134</td>
<td>0.286E-07</td>
<td>4.929</td>
</tr>
<tr>
<td>1.4</td>
<td>0.266E-04</td>
<td>0.768E-06</td>
<td>5.111</td>
<td>0.260E-07</td>
<td>4.884</td>
</tr>
<tr>
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<td>0.921E-04</td>
<td>0.644E-06</td>
<td>7.160</td>
<td>0.230E-07</td>
<td>4.809</td>
</tr>
<tr>
<td>1.8</td>
<td>0.173E-03</td>
<td>0.507E-06</td>
<td>8.416</td>
<td>0.198E-07</td>
<td>4.679</td>
</tr>
<tr>
<td>2.0</td>
<td>0.274E-03</td>
<td>0.361E-06</td>
<td>9.572</td>
<td>0.166E-07</td>
<td>4.438</td>
</tr>
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</table>

Table 5
Case 6: Errors and apparent convergence rates for the fully-discretization algorithm applied to the one-dimensional example with the choice of $N_x = 16, 32, 64, N_z = \frac{\pi}{16}, \epsilon = 0.1, \delta = 0.9$ and $\tau = \frac{1}{2\sqrt{N_x}} \approx 0.021$

<table>
<thead>
<tr>
<th>$N_z(N_x)$</th>
<th>16(16)</th>
<th>32(64)</th>
<th>$\rho_{12}$</th>
<th>64(256)</th>
<th>$\rho_{64}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.125E+00</td>
<td>0.157E-01</td>
<td>2.993</td>
<td>0.798E-03</td>
<td>4.300</td>
</tr>
<tr>
<td>0.4</td>
<td>0.102E+00</td>
<td>0.126E-01</td>
<td>3.022</td>
<td>0.602E-03</td>
<td>4.384</td>
</tr>
<tr>
<td>0.6</td>
<td>0.871E-01</td>
<td>0.105E-01</td>
<td>3.058</td>
<td>0.505E-03</td>
<td>4.373</td>
</tr>
<tr>
<td>0.8</td>
<td>0.838E-01</td>
<td>0.891E-02</td>
<td>3.233</td>
<td>0.422E-03</td>
<td>4.401</td>
</tr>
<tr>
<td>1.0</td>
<td>0.151E+00</td>
<td>0.787E-02</td>
<td>4.257</td>
<td>0.358E-03</td>
<td>4.458</td>
</tr>
<tr>
<td>1.2</td>
<td>0.241E+00</td>
<td>0.742E-02</td>
<td>5.021</td>
<td>0.311E-03</td>
<td>4.576</td>
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<tr>
<td>1.4</td>
<td>0.222E+00</td>
<td>0.786E-02</td>
<td>4.818</td>
<td>0.280E-03</td>
<td>4.813</td>
</tr>
<tr>
<td>1.6</td>
<td>0.195E+00</td>
<td>0.100E-01</td>
<td>4.284</td>
<td>0.264E-03</td>
<td>5.248</td>
</tr>
<tr>
<td>1.8</td>
<td>0.181E+00</td>
<td>0.171E-01</td>
<td>3.410</td>
<td>0.266E-03</td>
<td>6.006</td>
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<tr>
<td>2.0</td>
<td>0.209E+00</td>
<td>0.427E-01</td>
<td>2.292</td>
<td>0.288E-03</td>
<td>7.211</td>
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</tbody>
</table>

Table 6
Case 1: Errors and apparent convergence rates for the space-discretization algorithm applied to the two-dimensional example with the choice of $N_x = 128, N_z = 16, 32, 64, 128, \epsilon = 0.1, \delta = 0.9$ and $\tau = \frac{1}{2\sqrt{N_x}} \approx 0.289$

<table>
<thead>
<tr>
<th>$N_z$</th>
<th>16</th>
<th>32</th>
<th>$\rho_{12}$</th>
<th>64</th>
<th>$\rho_{64}$</th>
<th>128</th>
<th>$\rho_{128}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.153E-01</td>
<td>0.946E-02</td>
<td>0.692</td>
<td>0.814E-02</td>
<td>0.216</td>
<td>0.783E-02</td>
<td>0.056</td>
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<tr>
<td>0.4</td>
<td>0.654E-02</td>
<td>0.168E-02</td>
<td>1.959</td>
<td>0.489E-03</td>
<td>1.781</td>
<td>0.224E-03</td>
<td>1.129</td>
</tr>
<tr>
<td>0.6</td>
<td>0.602E-02</td>
<td>0.151E-02</td>
<td>1.995</td>
<td>0.387E-03</td>
<td>1.964</td>
<td>0.113E-03</td>
<td>1.776</td>
</tr>
<tr>
<td>0.8</td>
<td>0.625E-02</td>
<td>0.157E-02</td>
<td>1.997</td>
<td>0.391E-03</td>
<td>2.002</td>
<td>0.971E-04</td>
<td>2.010</td>
</tr>
<tr>
<td>1.0</td>
<td>0.672E-02</td>
<td>0.168E-02</td>
<td>1.997</td>
<td>0.421E-03</td>
<td>2.000</td>
<td>0.105E-03</td>
<td>2.005</td>
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<tr>
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<td>1.998</td>
<td>0.443E-03</td>
<td>1.999</td>
<td>0.111E-03</td>
<td>1.999</td>
</tr>
<tr>
<td>1.4</td>
<td>0.703E-02</td>
<td>0.176E-02</td>
<td>1.997</td>
<td>0.440E-03</td>
<td>1.999</td>
<td>0.110E-03</td>
<td>2.000</td>
</tr>
<tr>
<td>1.6</td>
<td>0.651E-02</td>
<td>0.163E-02</td>
<td>1.995</td>
<td>0.409E-03</td>
<td>1.999</td>
<td>0.102E-03</td>
<td>2.000</td>
</tr>
<tr>
<td>1.8</td>
<td>0.562E-02</td>
<td>0.141E-02</td>
<td>1.991</td>
<td>0.354E-03</td>
<td>1.998</td>
<td>0.085E-04</td>
<td>1.999</td>
</tr>
<tr>
<td>2.0</td>
<td>0.452E-02</td>
<td>0.114E-02</td>
<td>1.987</td>
<td>0.286E-03</td>
<td>1.997</td>
<td>0.715E-04</td>
<td>1.999</td>
</tr>
</tbody>
</table>

The elliptic problem is solved by piecewise linear element and the resulting $L^2$ errors are computed using a Gaussian quadrature for triangles with 13 points [2].

To begin with, we concentrate on $L^2$ spatial-discretization error parts only, neglecting the time-discretization errors for each time $t = 0.2, 0.4, \ldots, 2$. For this, motivated by the Tables 2 and 3, $N_z = 128$
Table 7  Errors and apparent convergence rates for the space-discretization algorithm applied to the two-dimensional example with the choice of $N_x = 128$, $N_y = 16, 32, 64, 128$, $\epsilon = 0.1$, $\delta = 0.9$ and $\tau = \frac{1}{\max N_y} = 0.289$

<table>
<thead>
<tr>
<th>$N_y$</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>$p_{12}$</th>
<th>128</th>
<th>$p_{128}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.119E-01</td>
<td>0.849E-02</td>
<td>0.483</td>
<td>0.790E-02</td>
<td>0.105</td>
<td>0.777E-02</td>
</tr>
<tr>
<td>0.4</td>
<td>0.568E-02</td>
<td>0.138E-02</td>
<td>2.042</td>
<td>0.333E-03</td>
<td>2.051</td>
<td>0.156E-03</td>
</tr>
<tr>
<td>0.6</td>
<td>0.615E-02</td>
<td>0.156E-02</td>
<td>1.976</td>
<td>0.418E-03</td>
<td>1.905</td>
<td>0.133E-03</td>
</tr>
<tr>
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<td>0.151E-02</td>
<td>1.998</td>
<td>0.378E-03</td>
<td>2.003</td>
<td>0.938E-04</td>
</tr>
<tr>
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<td>0.343E-03</td>
<td>2.000</td>
<td>0.854E-04</td>
</tr>
<tr>
<td>1.2</td>
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<td>0.117E-02</td>
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<td>0.292E-03</td>
<td>1.998</td>
<td>0.730E-04</td>
</tr>
<tr>
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<td>1.998</td>
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<tr>
<td>1.6</td>
<td>0.296E-02</td>
<td>0.745E-03</td>
<td>1.989</td>
<td>0.187E-03</td>
<td>1.997</td>
<td>0.467E-04</td>
</tr>
<tr>
<td>1.8</td>
<td>0.228E-02</td>
<td>0.575E-03</td>
<td>1.989</td>
<td>0.144E-03</td>
<td>1.997</td>
<td>0.360E-04</td>
</tr>
<tr>
<td>2.0</td>
<td>0.174E-02</td>
<td>0.438E-03</td>
<td>1.989</td>
<td>0.110E-03</td>
<td>1.997</td>
<td>0.274E-04</td>
</tr>
</tbody>
</table>

Table 8  Errors and apparent convergence rates for the full-discretization algorithm applied to the two-dimensional example with the choice of $N_x = N_y = 16, 32, 64, 128$, $\epsilon = 0.1$, $\delta = 0.9$ and $\tau = \frac{1}{\max N_y} \approx 0.289$

<table>
<thead>
<tr>
<th>$N_y$</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>$p_{32}$</th>
<th>128</th>
<th>$p_{128}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.225E-01</td>
<td>0.192E-01</td>
<td>0.234</td>
<td>0.136E-01</td>
<td>0.496</td>
<td>0.777E-02</td>
</tr>
<tr>
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<td>0.133E-03</td>
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<td>0.854E-04</td>
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<td>1.996</td>
<td>0.593E-04</td>
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<td>0.745E-03</td>
<td>1.984</td>
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<td>1.997</td>
<td>0.467E-04</td>
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<tr>
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<td>0.574E-03</td>
<td>1.986</td>
<td>0.144E-03</td>
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<td>0.360E-04</td>
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<td>0.110E-03</td>
<td>1.996</td>
<td>0.274E-04</td>
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</tbody>
</table>

Table 9  Errors and apparent convergence rates for the full-discretization algorithm applied to the two-dimensional example with the choice of $N_x = N_y = 16, 32, 64, 128$, $\epsilon = 0.1$, $\delta = 0.9$ and $\tau = \frac{1}{\max N_y} \approx 0.289$

<table>
<thead>
<tr>
<th>$N_y$</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>$p_{32}$</th>
<th>128</th>
<th>$p_{128}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.226E-01</td>
<td>0.192E-01</td>
<td>0.236</td>
<td>0.136E-01</td>
<td>0.497</td>
<td>0.777E-02</td>
</tr>
<tr>
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<td>0.686E-02</td>
<td>0.290E-02</td>
<td>1.242</td>
<td>0.819E-03</td>
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<td>0.158E-03</td>
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<tr>
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<td>0.381E-03</td>
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<td>0.128E-03</td>
</tr>
<tr>
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<td>0.572E-02</td>
<td>0.146E-02</td>
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<td>0.875E-04</td>
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<td>0.126E-02</td>
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<td>0.789E-04</td>
</tr>
<tr>
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<td>0.107E-02</td>
<td>1.988</td>
<td>0.268E-03</td>
<td>2.003</td>
<td>0.671E-04</td>
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<tr>
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<td>1.995</td>
<td>0.218E-03</td>
<td>1.997</td>
<td>0.545E-04</td>
</tr>
<tr>
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<td>0.694E-03</td>
<td>1.988</td>
<td>0.174E-03</td>
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<td>0.434E-04</td>
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<tr>
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<td>0.546E-03</td>
<td>1.991</td>
<td>0.137E-03</td>
<td>1.997</td>
<td>0.342E-04</td>
</tr>
<tr>
<td>2.0</td>
<td>0.170E-02</td>
<td>0.428E-03</td>
<td>1.989</td>
<td>0.107E-03</td>
<td>1.998</td>
<td>0.268E-04</td>
</tr>
</tbody>
</table>

frequency division is fixed and we increase the number of spatial mesh points. Let $e_{N_y}$ be the $L^2$ error when we divide space $N_x \times N_y$ times and $p_{N_y} = \log_2(e_{N_y}/e_{N_y})$. As is seen in Tables 6 and 7 the order of convergence is 2 as expected. By comparing Tables 6 and 7, we observe that the errors for $\mu = 1$ are smaller
than those for $\mu = 0$, which is observed frequently in our numerical experimentation; for example, Cases 4 and 6 can be compared to Cases 5 and 6. These phenomena are not explicitly stated in (4.14), and therefore (4.14) needs to be improved in a future study.

Finally, consider the full-discretization errors. To get optimal orders of convergence, we choose $N_s = CN_q^{1/2}$, as mentioned in Section 4.3, and $r = 2$ and $\tau = \frac{1}{\tan \theta_q}$. We then expect the order of convergence to be 2 for $t \geq 1$. Let $e_{N_s}$ be the $L^2$ error when we extract $N_s$ points on the contour in the frequency domain and divide the space domain with mesh size $1/N_t$, where $N_t = CN_q^{1/2}$ and $\rho_{N_s} = \log_2 (e_{N_{s/2}}/e_{N_s})$. In Tables 8 and 9 the convergence orders are nearly 2 for $t \geq 1$. Even for the time $t < 1$, the convergence order 2 is achieved, although they are not as good as for the case $t > 1$. The effect of $\beta$ can be observed in Tables 8 and 9. The $L^2$ errors for $\beta = 1$ are less than those for $\beta = 0$, although they are not so apparent as the effect of $\mu$ shown in Tables 6 and 7.

For Tables 10 and 11, the errors are computed with $N_s = CN_q^{1/2}$, $r = 3$ and $\tau = \frac{1}{3 \tan \theta_q}$. In this case we see convergence orders are approximately 3 for $t \geq 1$, as expected, moreover, convergence orders 3 can be achieved even for $t < 1$ with sufficiently enough contour points $N_s = 128$.

Besides the effects of $\mu$, $\alpha$, $\beta$, $r$, and $\tau$ observed above, the numerical solution of integro-differential equation depends on the parameters $\epsilon$ and $\delta$. The parameter $\epsilon$ defines the contour with the parameter $\theta_q$ and it is included in the constant $C$ in (4.8). More precisely speaking, if $\epsilon$ goes to 0 the constant $C$ in (4.8) tends

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### Table 10
Case 3: Errors and apparent convergence rates for the full-discretization algorithm applied to the two-dimensional example with the choice of $N_x = 16$, 32, 64, $N_s = \frac{1}{4} N_q^{1/2}$, $\epsilon = 0.1$, $\delta = 0.9$ and $\tau = \frac{1}{\tan \theta_q} \approx 0.192$

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>8(5)</th>
<th>16(16)</th>
<th>32(45)</th>
<th>64(128)</th>
<th>$\rho_{N_s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.129E+01</td>
<td>0.157E-01</td>
<td>6.357</td>
<td>0.226E-02</td>
<td>2.793</td>
</tr>
<tr>
<td>0.4</td>
<td>0.105E+01</td>
<td>0.640E-02</td>
<td>7.362</td>
<td>0.951E-03</td>
<td>2.751</td>
</tr>
<tr>
<td>0.6</td>
<td>0.862E+00</td>
<td>0.546E-02</td>
<td>7.302</td>
<td>0.687E-03</td>
<td>2.992</td>
</tr>
<tr>
<td>0.8</td>
<td>0.706E+00</td>
<td>0.531E-02</td>
<td>7.055</td>
<td>0.673E-03</td>
<td>2.980</td>
</tr>
<tr>
<td>1.0</td>
<td>0.578E+00</td>
<td>0.541E-02</td>
<td>6.739</td>
<td>0.687E-03</td>
<td>2.787</td>
</tr>
<tr>
<td>1.2</td>
<td>0.473E+00</td>
<td>0.549E-02</td>
<td>6.429</td>
<td>0.699E-03</td>
<td>2.975</td>
</tr>
<tr>
<td>1.4</td>
<td>0.387E+00</td>
<td>0.533E-02</td>
<td>6.183</td>
<td>0.682E-03</td>
<td>2.966</td>
</tr>
<tr>
<td>1.6</td>
<td>0.317E+00</td>
<td>0.490E-02</td>
<td>6.016</td>
<td>0.632E-03</td>
<td>2.954</td>
</tr>
<tr>
<td>1.8</td>
<td>0.260E+00</td>
<td>0.426E-02</td>
<td>5.930</td>
<td>0.557E-03</td>
<td>2.935</td>
</tr>
<tr>
<td>2.0</td>
<td>0.213E+00</td>
<td>0.350E-02</td>
<td>5.924</td>
<td>0.468E-03</td>
<td>2.904</td>
</tr>
</tbody>
</table>

---

### Table 11
Case 5: Errors and apparent convergence rates for the full-discretization algorithm applied to the two-dimensional example with the choice of $N_x = 16$, 32, 64, $N_s = \frac{1}{4} N_q^{1/2}$, $\epsilon = 0.1$, $\delta = 0.9$ and $\tau = \frac{1}{\tan \theta_q} \approx 0.028$

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>8(5)</th>
<th>16(16)</th>
<th>32(45)</th>
<th>64(128)</th>
<th>$\rho_{N_s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.129E+01</td>
<td>0.419E+00</td>
<td>1.619</td>
<td>0.674E-01</td>
<td>2.636</td>
</tr>
<tr>
<td>0.4</td>
<td>0.105E+01</td>
<td>0.348E+00</td>
<td>1.595</td>
<td>0.636E-01</td>
<td>2.454</td>
</tr>
<tr>
<td>0.6</td>
<td>0.862E+00</td>
<td>0.283E+00</td>
<td>1.605</td>
<td>0.532E-01</td>
<td>2.414</td>
</tr>
<tr>
<td>0.8</td>
<td>0.706E+00</td>
<td>0.249E+00</td>
<td>1.505</td>
<td>0.445E-01</td>
<td>2.484</td>
</tr>
<tr>
<td>1.0</td>
<td>0.578E+00</td>
<td>0.245E+00</td>
<td>1.240</td>
<td>0.383E-01</td>
<td>2.675</td>
</tr>
<tr>
<td>1.2</td>
<td>0.473E+00</td>
<td>0.302E+00</td>
<td>0.649</td>
<td>0.339E-01</td>
<td>3.155</td>
</tr>
<tr>
<td>1.4</td>
<td>0.387E+00</td>
<td>0.897E-00</td>
<td>-1.212</td>
<td>0.320E-01</td>
<td>4.808</td>
</tr>
<tr>
<td>1.6</td>
<td>0.317E+00</td>
<td>0.115E+00</td>
<td>-1.856</td>
<td>0.332E-01</td>
<td>5.112</td>
</tr>
<tr>
<td>1.8</td>
<td>0.260E+00</td>
<td>0.105E+01</td>
<td>-2.020</td>
<td>0.389E-01</td>
<td>4.757</td>
</tr>
<tr>
<td>2.0</td>
<td>0.213E+00</td>
<td>0.918E+00</td>
<td>-2.110</td>
<td>0.519E-01</td>
<td>4.143</td>
</tr>
</tbody>
</table>
to increase if $\beta = 0$. Thus too small $\epsilon + \beta$ may result in relatively large numerical error. Also if $\epsilon$ is too large, the computational errors will be large as seen in the term $e^{t+c\cot \theta}$ in (4.14). We therefore have chosen the parameter $\epsilon = 0.1$ in our numerical implementations. The parameter $\delta$ also affects the upper bound of $\|E(z)\|$ in (4.8). But the effect does not seem to be critical compared to other parameters.

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References